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New Results for Time-Optimal Three-Axis Reorientation of a Rigid Spacecraft

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Some new results for a classical minimum-time rest-to-rest maneuver problem are presented. An inertially symmetric rigid body is considered. For the case in which the magnitude of the control is constrained while the control direction is left free, we analytically prove that the eigenaxis maneuver is the time-minimum solution by using Pontryagin's principle. For the case in which the three components of the control are independently constrained, we discover six- and seven-switch solutions for reorientation angles of less than 72 deg by using a hybrid numerical approach. The seven-switch solutions are consistent with the classical results, and the six-switch solutions are reported here for the first time. We find that the two sets of solutions are widely separated in state and control spaces. However, the two locally optimum maneuver times are very close to each other, although the six-switch control always has a slightly shorter time. Simulation results that illustrate and validate the new findings are summarized. Although no conclusive proof is available to date, we believe the six-switch maneuvers to be global extrema.

I. Introduction

MULTI-AXIS reorientation of a spacecraft in minimum time is a fundamentally interesting problem from both practical and mathematical points of view. Space operations very often involve transferring the spacecraft from one attitude to another in minimum time. In such cases, usually there will be some constraints on the amount of control torque that can be applied. Theoretically, the first-order necessary optimality conditions for this maneuver yield a two-point boundary value problem. Because the control is appearing linearly in the dynamic equations, Pontryagin's principle [1] leads to a switching-type controller. For multi-axis attitude and angular velocity optimal control, nonlinear equations of motion are involved. No rigorous analytical solution for a general time-optimal maneuver has been published so far.

A particular motivation for this paper is the results that Bilimoria and Wie [2] obtained for a simplified inertially symmetric body. They used a quaternion parameterization of attitude and considered the minimum-time reorientation. They proposed admissible control constraint conditions such that each of the three orthogonal control components is less than a specified value. They found the solution of the two-point boundary value problem using a trial-and-error process and an unspecified continuation approach. Furthermore, their results indicated that, for a reorientation angle of less than 72 deg, the optimal control involved seven switches at distinct switch times. These results may look counterintuitive at first glance as the eigenaxis maneuver seems a natural solution to the minimum-time maneuver problem for this symmetric body [3–5].

One goal of this paper is to show that whether eigenaxis maneuver is optimal or not depends on the definition of the set of admissible control. When the three orthogonal components of the control are independently constrained, the noneigenaxis maneuvers that Bilimoria and Wie found require less maneuver time because the nutational components can provide more torques along the reorientation axis [2]. However, when the total magnitude of the control vector is constrained, the eigenaxis maneuver is indeed the time-optimal solution. This will be proved in Sec. III.

We have also revisited the solutions reported in Bilimoria and Wie [2]. The issue of the global optimality of their solutions appears to be important because, based on our literature review study, their results have been a baseline for several subsequent papers [6–9]. Byers and Vadali [6] considered the case in which the applied torque is much greater than the nonlinear terms in Euler's equation. Approximate solutions are derived to analyze switch times. They compared their analytical solutions with Bilimoria and Wie's. Scrivener and Thompson [7] used collocation and nonlinear programming methods to solve the same problem. Their results were consistent with Bilimoria and Wie's until the reorientation angles were less than 10 deg. For angles of less than 10 deg, they found 11 switches. The authors conjectured that they may not have found valid solutions for the new control structure and that their new solutions may have been the result of numerical difficulties because of the small angles. No conclusions are given in the paper.

These previous studies motivated us to answer some key questions: Are the solutions reported by Bilimoria and Wie the global optimal solutions? If not, are there other local extrema? Theoretically, for this nonlinear problem, although the necessary conditions Bilimoria and Wie used yield a local stationary point, there is no guarantee that their numerical search led to a unique stationary solution or the global minimum-time maneuver. When we study this fundamental minimum-time maneuver problem, we find an apparently heretofore unknown truth that there are indeed multiple stationary solutions (at least two) that satisfy the necessary conditions: Bilimoria and Wie's original solution and, more important, a six-switch local minimum-time maneuver for which the maneuver time is slightly shorter than the now-classical result of Bilimoria and Wie. We report here the main results of our investigation: We have confirmed Bilimoria and Wie's solution and we have also found a second local minimum. Our results indicate that Bilimoria and Wie's solution is not the global minimum. We show that the two locally optimal maneuvers are significantly different in both the control space and the state space; however, the maneuver times for these two local extrema are very close.

After obtaining the two sets of solutions, we noticed that there are very few published results on the second-order sufficient conditions for bang–bang control problems. The pioneering studies by Sarychev [10] and the recent work by Agrachev et al. [11] did not provide a clear approach for implementation and do not appear attractive for numerical validation. Maurer and Osmolovskii [12] provided a systematic numerical method to verify the second-order conditions. In the case of one or two switches, the tests are very easy

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to implement. However, Maurer and Osmolovskii precluded simultaneous switching bang–bang control structures when they derived the second-order conditions. We have verified that the seven-switch maneuver satisfies the second-order conditions proposed in Maurer and Osmolovskii's paper. However, for the new control structure using six switches, our results show that there are two control components switching simultaneously. This multiple simultaneous switch phenomenon happens for all the maneuver solutions when the reorientation angle is less than 72 deg. Thus, the conditions reported by Maurer and Osmolovskii could not be directly applied to the present problem.

A major difference between the way we formulate the problem and most of the previous minimum-time maneuver studies is the kinematic description. It has become common to use Euler parameters (the elements of the quaternion) instead of Euler angles for attitude descriptions because of the nonsingular behavior of the kinematic description [13]. However, even though Vadali [14] has rigorously shown that the unit norm constraint of the Euler parameters does not need to be included explicitly in the formulation of the system, redundant parameters bring extra costates to be solved. In this paper, we use the modified Rodrigues parameter (MRP) vector [15] instead of the Euler parameters for the kinematic description because the MRP vector is a nonredundant three-parameter description. When iterating to solve the two-point boundary value problem, the advantages of minimal parameter representation are obvious. Using the MRP vector reduces the dimension of this two-point boundary value problem from 14 to 12. Additionally, we find that, by using the MRP vector, it is easy to extract the initial costates for the hybrid approach that we use in this paper. Also notice that the MRP vector has no geometric singularity in the closed interval of rotational angle ± 360 deg, which is the range of maneuvers for most spacecraft practical applications and all of those considered in this paper.

We now provide an overview of the structure of the present paper. In Sec. II, we use the MRP vector to formulate the optimal control problem for a spherically symmetric rigid body. In Sec. III, we prove that the eigenaxis maneuver is the time-optimal solution when the control is constrained within a sphere. In Sec. IV, the numerical approach to solve the optimal control for the case in which the control is within a cube is presented, followed by the simulation results that show the differences between the two locally optimal maneuvers. We present our conclusions in Sec. V.

II. Problem Formulation

The nondimensional rotational equation for the spherically symmetric rigid body is formulated by Bilimoria and Wie [2] as

$$\dot{\boldsymbol{\omega}} = \boldsymbol{u}, \qquad \boldsymbol{\omega} \in R^3, \boldsymbol{u} \in R^3 \tag{1}$$

where $\boldsymbol{\omega}$ is the nondimensional angular velocity, and $\boldsymbol{u}$ is the nondimensional control.

Using the MRP vector, the kinematic equation is [15]

$$\begin{aligned}\dot{\boldsymbol{\sigma}} &= 1/4[B(\boldsymbol{\sigma})]\boldsymbol{\omega} \\ &= 1/4\begin{bmatrix} 1-\sigma^2+2\sigma_1^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma^2+2\sigma_2^2 & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma^2+2\sigma_3^2 \end{bmatrix}\boldsymbol{\omega}\end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}^{\mathsf{T}} = [\sigma_1, \sigma_2, \sigma_3]^{\mathsf{T}}$, and

$$\sigma^2 = \sum_{i=1}^{3} \sigma_i^2 \tag{3}$$

The MRP vector is related to the principal rotation angle ϕ and eigenvector $\hat{\boldsymbol{e}}$ by [15]

$$\boldsymbol{\sigma} = \tan(\phi/4)\hat{\boldsymbol{e}} \tag{4}$$

Consider a rest-to-rest maneuver with a net reorientation about the inertial Iz axis. The principal rotation angle is ϕ_f. The boundary conditions for the angular velocity and the MRP vector are

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = 0 \tag{5}$$

$$\sigma_1(0) = \sigma_2(0) = \sigma_3(0) = 0 \tag{6}$$

$$\omega_1(T_f) = \omega_2(T_f) = \omega_3(T_f) = 0 \tag{7}$$

$$\sigma_1(T_f) = \sigma_2(T_f) = 0; \qquad \sigma_3(T_f) = \tan(\phi/4) \tag{8}$$

where we assume the initial time is 0 and the final time is T_f. The Hamiltonian for minimizing the maneuvering time is

$$\begin{aligned}H &= 1 + \lambda_{\omega_1}u_1 + \lambda_{\omega_2}u_2 + \lambda_{\omega_3}u_3 + \lambda_{\sigma_1}\dot{\sigma_1} + \lambda_{\sigma_2}\dot{\sigma_3} + \lambda_{\sigma_3}\dot{\sigma_3} \\ &= 1 + \boldsymbol{\lambda}_\omega^{\mathsf{T}}\boldsymbol{u} + \boldsymbol{\lambda}_\sigma^{\mathsf{T}}\dot{\boldsymbol{\sigma}}\end{aligned} \tag{9}$$

To satisfy the first-order necessary conditions [1], the differential equations for the costates are

$$\begin{aligned}\dot{\lambda}_{\omega_1}(t) &= -\frac{\partial H}{\partial \omega_1} = -\lambda_{\sigma_1}\left(\tfrac{1}{4}+\tfrac{1}{4}\sigma_1^2-\tfrac{1}{4}\sigma_2^2-\tfrac{1}{4}\sigma_3^2\right) \\ &\quad - \lambda_{\sigma_2}\left(\tfrac{1}{2}\sigma_1\sigma_2+\tfrac{1}{2}\sigma_3\right) - \lambda_{\sigma_3}\left(\tfrac{1}{2}\sigma_1\sigma_3-\tfrac{1}{2}\sigma_2\right)\end{aligned} \tag{10}$$

$$\begin{aligned}\dot{\lambda}_{\omega_2}(t) &= -\frac{\partial H}{\partial \omega_2} = -\lambda_{\sigma_1}\left(\tfrac{1}{2}\sigma_1\sigma_2-\tfrac{1}{2}\sigma_3\right) \\ &\quad - \lambda_{\sigma_2}\left(\tfrac{1}{4}-\tfrac{1}{4}\sigma_1^2+\tfrac{1}{4}\sigma_2^2-\tfrac{1}{4}\sigma_3^2\right) - \lambda_{\sigma_3}\left(\tfrac{1}{2}\sigma_2\sigma_3+\tfrac{1}{2}\sigma_1\right)\end{aligned} \tag{11}$$

$$\begin{aligned}\dot{\lambda}_{\omega_3}(t) &= -\frac{\partial H}{\partial \omega_3} = -\lambda_{\sigma_1}\left(\tfrac{1}{2}\sigma_1\sigma_3+\tfrac{1}{2}\sigma_2\right) - \lambda_{\sigma_2}\left(\tfrac{1}{2}\sigma_2\sigma_3-\tfrac{1}{2}\sigma_1\right) \\ &\quad - \lambda_{\sigma_3}\left(\tfrac{1}{4}-\tfrac{1}{4}\sigma_1^2-\tfrac{1}{4}\sigma_2^2+\tfrac{1}{4}\sigma_3^2\right)\end{aligned} \tag{12}$$

$$\begin{aligned}\dot{\lambda}_{\sigma_1}(t) &= -\frac{\partial H}{\partial \sigma_1} = -\lambda_{\sigma_1}\left(\tfrac{1}{2}\sigma_1\omega_1+\tfrac{1}{2}\sigma_2\omega_2+\tfrac{1}{2}\sigma_3\omega_3\right) \\ &\quad - \lambda_{\sigma_2}\left(\tfrac{1}{2}\sigma_2\omega_1-\tfrac{1}{2}\sigma_1\omega_2-\tfrac{1}{2}\omega_3\right) - \lambda_{\sigma_3}\left(\tfrac{1}{2}\sigma_3\omega_1+\tfrac{1}{2}\omega_2-\tfrac{1}{2}\sigma_1\omega_3\right)\end{aligned} \tag{13}$$

$$\begin{aligned}\dot{\lambda}_{\sigma_2}(t) &= -\frac{\partial H}{\partial \sigma_2} = -\lambda_{\sigma_1}\left(-\tfrac{1}{2}\sigma_2\omega_1+\tfrac{1}{2}\sigma_1\omega_2+\tfrac{1}{2}\omega_3\right) \\ &\quad - \lambda_{\sigma_2}\left(\tfrac{1}{2}\sigma_1\omega_1+\tfrac{1}{2}\sigma_2\omega_2+\tfrac{1}{2}\sigma_3\omega_3\right) \\ &\quad - \lambda_{\sigma_3}\left(-\tfrac{1}{2}\omega_1+\tfrac{1}{2}\sigma_3\omega_2-\tfrac{1}{2}\sigma_2\omega_3\right)\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{\lambda}_{\sigma_3}(t) &= -\frac{\partial H}{\partial \sigma_3} = -\lambda_{\sigma_1}\left(-\tfrac{1}{2}\sigma_3\omega_1-\tfrac{1}{2}\omega_2+\tfrac{1}{2}\sigma_1\omega_3\right) \\ &\quad - \lambda_{\sigma_2}\left(\tfrac{1}{2}\omega_1-\tfrac{1}{2}\sigma_3\omega_2+\tfrac{1}{2}\sigma_2\omega_3\right) - \lambda_{\sigma_3}\left(\tfrac{1}{2}\sigma_1\omega_1+\tfrac{1}{2}\sigma_2\omega_2+\tfrac{1}{2}\sigma_3\omega_3\right)\end{aligned} \tag{15}$$

For this minimum-time problem, the boundary condition [1] requires the Hamiltonian in Eq. (9) to be zero at the final time. Depending on which of the two different types of control constraints is chosen, we will have two different optimal solutions, as shown in the following two sections.

III. Admissible Control Torque Vector Lies Within a Unit Sphere

When the magnitude of the control is constrained while the control direction is left free, a body-fixed coordinate system can be defined such that the body z axis is aligned with the inertial Iz axis, which is the eigenaxis for the reorientation maneuver defined in Sec. II. The body x and y axes are left free but are required to complete a right-hand orthogonal reference frame. As shown in Fig. 1, two angles, α and β, are used to define the control direction relative to the specified reference frame. For the case in which the magnitude of the nondimensional control is constrained by one, the three control components in the prescribed frame are

$$\boldsymbol{L}_c(t) = \begin{bmatrix} L(t)\sin(\alpha(t)) \\ L(t)\cos(\alpha(t))\cos(\beta(t)) \\ L(t)\cos(\alpha(t))\sin(\beta(t)) \end{bmatrix} \tag{16}$$

where $0 \le L(t) \le 1$, $-(\pi/2) \le \alpha(t) \le (\pi/2)$, and $0 \le \beta(t) < 2\pi$.

Using the first- and second-order optimality conditions [1], the optimal control angles are

$$\alpha = a\tan 2\left(-\lambda_{\omega_1}, \sqrt{\lambda_{\omega_2}^2 + \lambda_{\omega_3}^2}\right) \tag{17}$$

$$\beta = a\tan 2(-\lambda_{\omega_3}, -\lambda_{\omega_2}) \tag{18}$$

where $a\tan 2(\cdot)$ is the four-quadrant inverse tangent function that returns the angles in the interval $[-\pi/2, \pi/2]$.

The optimal control magnitude is

$$L = 0, \quad \text{if } \lambda_{\omega_1}\sin(\alpha) + \lambda_{\omega_2}\cos(\alpha)\cos(\beta) + \lambda_{\omega_3}\cos(\alpha)\sin(\beta) > 0 \tag{19}$$

$$L = 1, \quad \text{if } \lambda_{\omega_1}\sin(\alpha) + \lambda_{\omega_2}\cos(\alpha)\cos(\beta) + \lambda_{\omega_3}\cos(\alpha)\sin(\beta) \le 0 \tag{20}$$

Define the initial conditions for the costates as

$$\lambda_{\omega_1}(0) = 0 \tag{21}$$

$$\lambda_{\omega_2}(0) = 0 \tag{22}$$

$$\lambda_{\omega_3}(0) = -1 \tag{23}$$

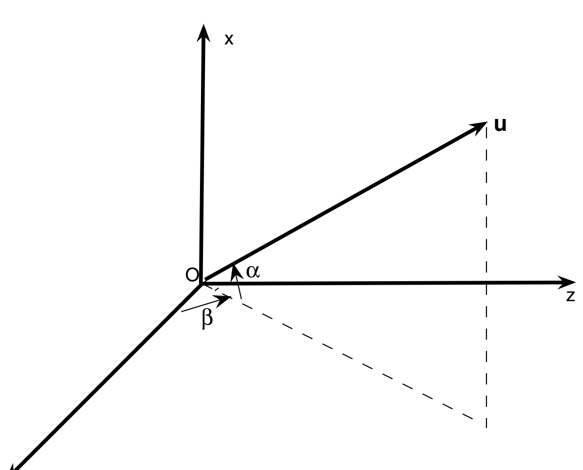


Fig. 1 Control direction definition.

$$\lambda_{\sigma_1}(0) = 0 \tag{24}$$

$$\lambda_{\sigma_2}(0) = 0 \tag{25}$$

$$\lambda_{\sigma_3}(0) = -(4/\sqrt{\phi_f}) \tag{26}$$

The analytical solutions for this initial value problem are shown in the following equations with the minimum-time solution $T_f = 2\sqrt{\phi_f}$.

The angular velocity history is

$$\omega_1(t) = 0 \tag{27}$$

$$\omega_2(t) = 0 \tag{28}$$

$$\omega_3(t) = t, \qquad 0 \le t \le \sqrt{\phi_f}; \qquad \omega_3(t) = -t + 2\sqrt{\phi_f} \qquad \sqrt{\phi_f} < t < 2\sqrt{\phi_f} \tag{29}$$

The MRP vector time history follows:

$$\sigma_1(t) = 0 \tag{30}$$

$$\sigma_2(t) = 0 \tag{31}$$

$$\sigma_3(t) = \text{actan}\left(\tfrac{1}{8}t^2\right), \qquad 0 \le t \le \sqrt{\phi_f};$$
$$\sigma_3(t) = \text{actan}\left(-\tfrac{1}{8}t^2 + (\sqrt{\phi}/2)t - \tfrac{1}{4}\phi_f\right), \qquad \sqrt{\phi_f} < t < 2\sqrt{\phi_f} \tag{32}$$

The costate histories are

$$\lambda_{\omega_1}(t) = 0 \tag{33}$$

$$\lambda_{\omega_2}(t) = 0 \tag{34}$$

$$\lambda_{\omega_3}(t) = -1 + (1/\sqrt{\phi_f})t \tag{35}$$

and

$$\lambda_{\sigma_1}(t) = 0 \tag{36}$$

$$\lambda_{\sigma_2}(t) = 0 \tag{37}$$

$$\dot{\lambda}_{\sigma_3} = -\tfrac{1}{2}\lambda_{\sigma_3}\sigma_3\omega_3 \tag{38}$$

The optimal control law from Eqs. (17) and (18) provides the directional angles:

$$\alpha = 0 \tag{39}$$

$$\beta = a\tan 2(-\lambda_{\omega_3}, 0) = \pi/2, \qquad 0 \le t \le \sqrt{\phi_f};$$
$$\beta = a\tan 2(-\lambda_{\omega_3}, 0) = -\pi/2, \qquad \sqrt{\phi_f} \le t \le 2\sqrt{\phi_f} \tag{40}$$

These control angle solutions, together with Eqs. (19) and (20), lead to the optimal control magnitude as $L = 1$. The Hamiltonian at the final time is zero because

$$H(T_f) = H(2\sqrt{\phi_f})$$
$$= 1 + \lambda_{\omega_3}(2\sqrt{\phi_f})u_3(2\sqrt{\phi_f}) + \lambda_{\sigma_3}(2\sqrt{\phi_f})\dot{\sigma}_3(2\sqrt{\phi_f})$$
$$= 1 + (-1) \times (1) + 0 = 0 \tag{41}$$

Equations (27–32) satisfy the boundary conditions $\omega_1(T_f) = \omega_2(T_f) = \omega_3(T_f) = 0$, $\sigma_1(T_f) = \sigma_2(T_f) = 0$, and $\sigma_3(T_f) = \tan(\frac{\phi_f}{4})$. They also show that the optimal motion is an eigenaxis maneuver. Equations (39–41) show that the optimal control is a bang–bang control about the eigenaxis.

IV. Admissible Control Torque Vector Lies Within a Unit Cube

A. Optimal Control Formulation

This is the case that was discussed by Bilimoria and Wie [2]. The bounds on the admissible control components are

$$-1 \le u_i(t) \le 1, \qquad i = 1, 2, 3 \tag{42}$$

By invoking Pontryagin's principle [1], the optimal control law from minimization of H in Eq. (9) over the admissible set of Eq. (42) has the form

$$\boldsymbol{u} = -\text{sign}(\boldsymbol{\lambda}_\omega) \tag{43}$$

or, explicitly,

$$u_i = 1, \quad \text{if } \lambda_\omega(i) < 0 \tag{44}$$

$$u_i = -1, \quad \text{if } \lambda_\omega(i) > 0 \tag{45}$$

$$u_i = u_s, \quad \text{if } \lambda_\omega(i) = 0 \tag{46}$$

where $i = \{1, 2, 3\}$, and u_s represents some singular control law. We point out that the singular control case in which one or two components of the costates $\boldsymbol{\lambda}_\omega$ becomes zero over a finite time interval is not formally considered in this paper. Both Bilimoria and Wie's paper [2] and the present paper found no numerical evidence that singular subarcs exist for this problem, but a rigorous proof is not available so far.

B. Hybrid Approach to Solve the Problem

To improve the accuracy of the direct optimization solutions and to enlarge the convergence domain of the indirect methods, Stryk and Bulirsch [16] proposed a hybrid approach to solve the optimal control problem. This cascaded computational scheme has become widely applied in many recent papers [9,17]. The key idea is to extract the costates and other control structure information from a nonlinear programming approach as a first step. The indirect shooting method is then used to refine the solutions. We summarize the three major steps we use to solve for the optimal maneuver solutions and to validate the results based on the first-order optimality conditions [1].

Step 1: The kinematic and dynamic differentiation equations are discretized using the trapezoidal method [18]. Function fmincon in MATLAB® is used to get the preliminary and approximate control structures, switching times, and initial costates.

Step 2: Using the results from step 1 as the initial guess, fmincon is used as a shooting method to solve the two-point boundary value problem. The constraints include the final time conditions and the invariance of the Hamiltonian.

Step 3: The results from step 2, together with the originally known initial time state conditions, are used to solve for the dynamic system response by integrating the kinematic and dynamic equations forward in time. The Hamiltonian history and the final state errors are the validation criteria.

We find that this three-step process efficiently leads to accurately converged results. However, we can only guarantee that the solutions we find are local extrema. By studying various experiments, we are fairly confident, however, that there are just two sets of local extrema for this problem and that the new results reported herein give the global minimum-time maneuver.

C. Numerical Results and Validation

We find five-switch solutions when the reorientation angles are greater than 72 deg, and our solutions are consistent with the results published in [2]. We find seven- and six-switch maneuvers when the reorientation angles are less than or equal to 72 deg. The final boundary condition errors are less than 10^{-12} for all the following maneuver solutions. The interesting discovery is that, for all reorientation angles less than 72 deg, the two maneuvers are totally distinct even though the optimal times for the two maneuver times are nearly equal. For example, when the reorientation angle is 45 deg, the nondimensional maneuver time [2] is 1.7499 for the seven-switch case and 1.7471 for the six-switch case. The angular velocity, Hamiltonian, MRP vector, and control structure for the two sets of maneuvers with a 45 deg reorientation are shown in Figs. 2–7. Comparing Figs. 2 and 5, we can see that the dynamic responses for these two controls are totally different. Figures 3, 4, 6, and 7 confirm

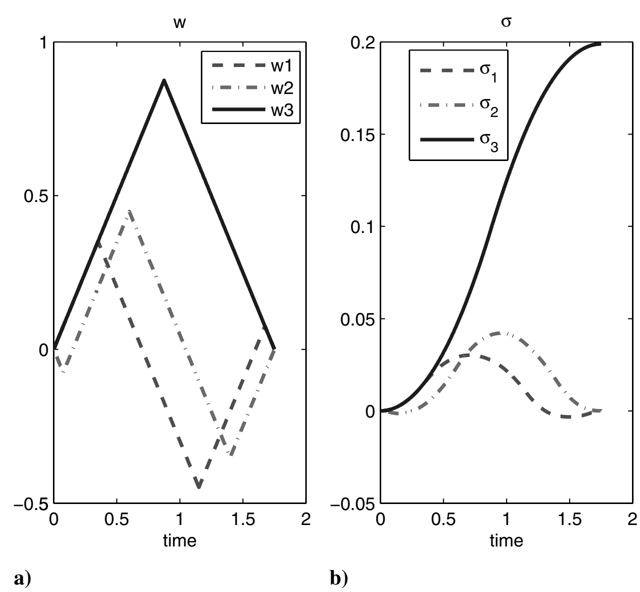


Fig. 2 45 deg maneuver with seven switches: a) angular velocity, and b) MRP.

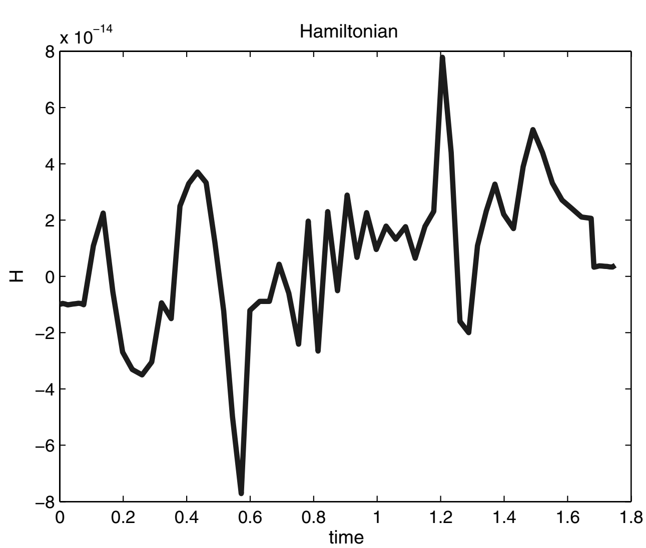


Fig. 3 45 deg maneuver with seven switches: Hamiltonian.

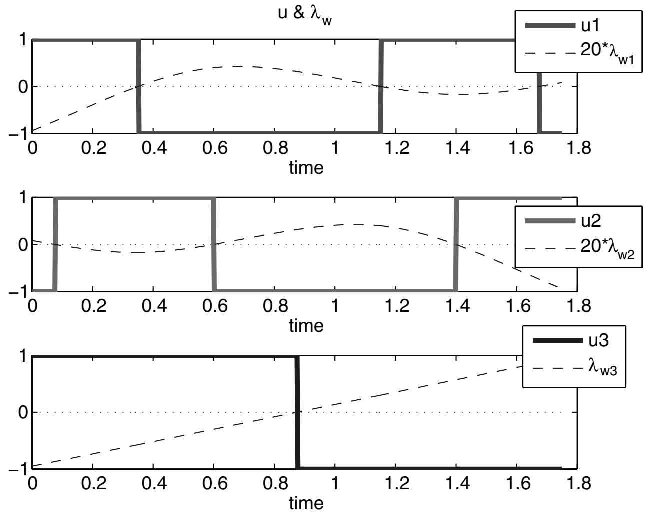
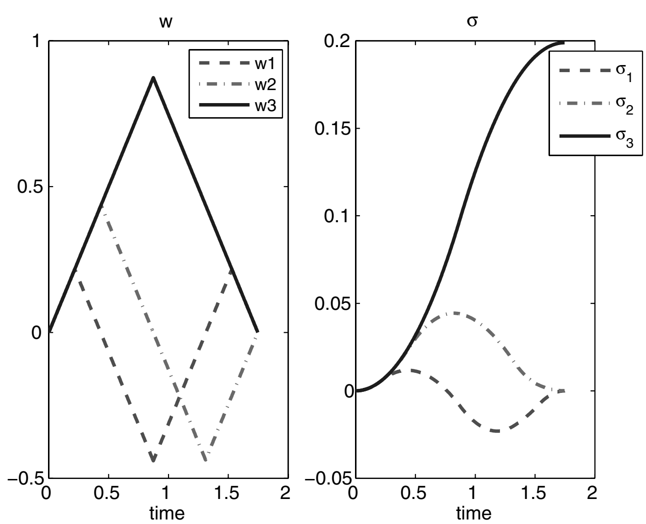


Fig. 4 45 deg maneuver with seven switches: control with λ_ω.

that both control laws satisfy Pontryagin's principle. As shown in Fig. 7, for the six-switch case, both u_1 and u_3 switch in the middle of the maneuver. We find that this simultaneous switch pattern holds for all six solutions when the reorientation angle runs from 72 to 1 deg.

Figure 8 shows the relative maneuver time increase by using a seven-switch control instead of a six-switch control. The sample reorientation angles are chosen as $\{1, 10, 45, 60, 72\}$ deg. We can see that the superiority of the six-switch control over the seven-switch control is most significant for the 45 deg reorientation. Figure 9 shows the contour plot of rotational displacement difference between the two local minimum solutions as a function of both the maneuver angle and resulting maneuver time. The principle rotation angles are calculated from the MRP vector using the forms given in Schaub and Junkins [15]. For a specific degree of reorientation, the principle rotation angles for the two maneuvers are the same initially. Their difference increases until halfway through the maneuver, corresponding to half of the minimum time. The difference then decreases back to zero at the final time, which shows the consistency of the boundary conditions. For different reorientation maneuvers, the maximum difference appears for the 72 deg maneuver. The maximum principle rotation angle difference is about 18 deg.



a) b)

Fig. 5 45 deg maneuver with six switches: a) angular velocity, and b) MRP.

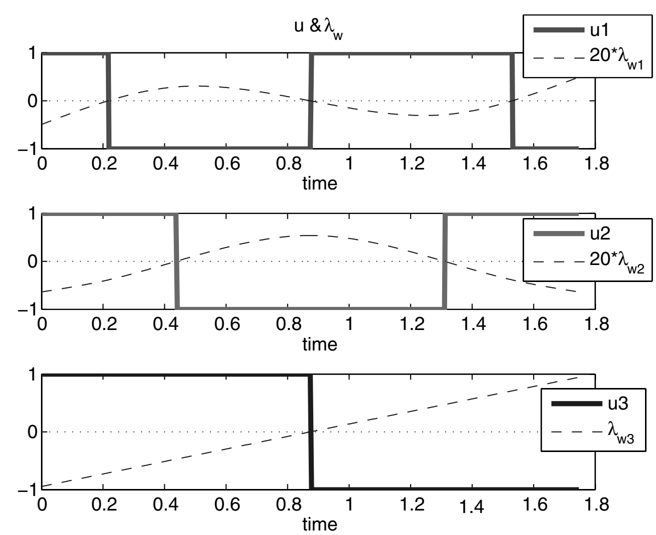


Fig. 7 45 deg maneuver with six switches: control with λ_ω.

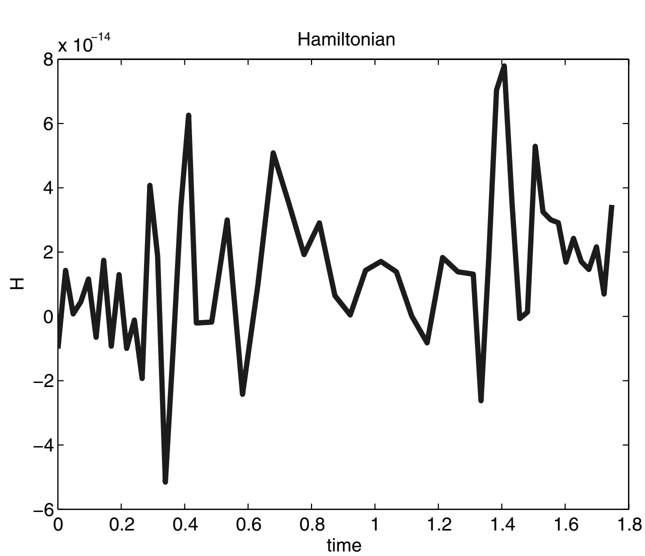


Fig. 6 45 deg maneuver with six switches: Hamiltonian.

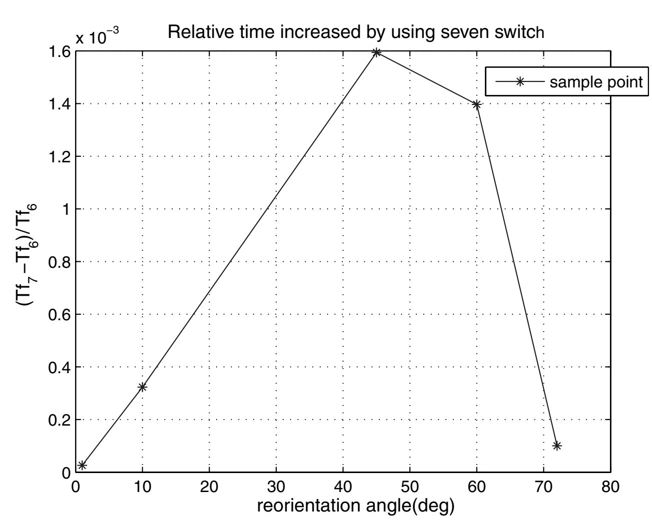


Fig. 8 Relative time increase by using the classical seven-switch local extrema $(Tf_7)^2$ compared with the new solutions using the six-switch local extrema (Tf_6).

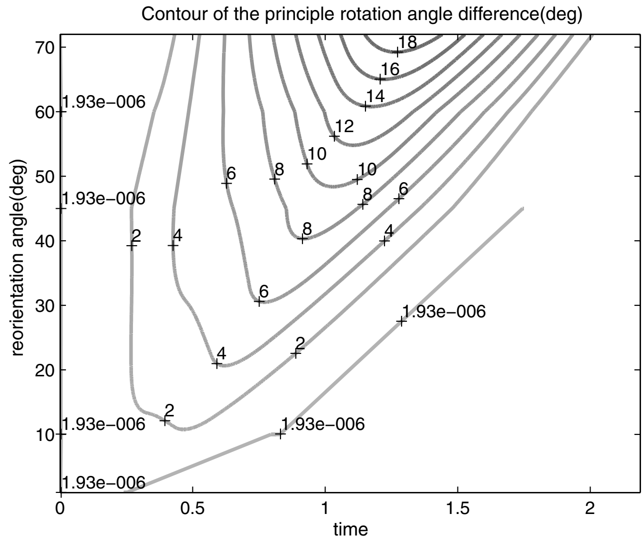


Fig. 9 Contour of the difference of the principle rotation angle: ϕ.

V. Conclusions

Some new results for the classical minimum-time rest-to-rest inertially symmetric rigid body maneuver problem are presented and verified. The intuitive eigenaxis maneuver is proved to be the optimal minimum-time solution when the control is constrained within a unit sphere. When the control components are independently constrained within a unit cube, we show that Bilimoria and Wie's seven-switch control laws and our new control laws with six switches are distinct local optimal solutions. After we obtained these new findings, we also resorted to a different optimizer, SNOPT 7.2 [19], to solve for the initial costates for further validation. The solutions returned by SNOPT agree with high precision with the results obtained using fmincon. We also discretized the system and solved the transcribed nonlinear programming problem ("direct" minimization) using SNOPT. We find that, depending on the initial guesses to the solution, SNOPT may converge locally to either the six- or seven-switch solution, again confirming the existence of local minima.

Although the maneuver times for the new six-switch local optima are slightly shorter than for Bilimoria and Wie's seven-switch local optima and are, we believe, the global optimal solutions, we observe that the reduction in maneuver time is small. For the case in which the reorientation angle is 45 deg, the time increase by using seven switches instead of six switches is about 0.15%. We did one test to eliminate one switch out of the six switches; the solution violated Pontryagin's principle in this case but the maneuver time increased by only 0.39%. Considering the time increase by using the eigenaxis maneuver, which only involves one switch, is only 1.45% longer, we think (as a practical matter) that all these interesting numbers tell us that the easy-to-implement eigenaxis control remains a very good suboptimal control solution. We also point out that, for both the seven- and six-switch solutions, there exists a finite number of control patterns that can be obtained by permutation of the control structures given in this paper. Obviously, the number of switches will not change as a result of the permutation, but the suboptimal state space excursions and variations in maneuver time may be of interest. Finally, by revisiting this classical problem, we find there remain some interesting nonlinear behaviors involved with this problem that are not yet well understood. We need further qualitatively insights: Why is the five-switch control the optimal solution for reorientation angles greater than 72 deg? Why are there two local solutions with six or seven switches when the reorientation angle is equal to or less than 72 deg? Is there an easy-to-implement second-order optimality condition for a general bang–bang control problem besides the current results of Maurer and Osmolovskii?

Even though these questions remain, this paper, together with the cited literature, brings the solution of this classical problem to a mature state of completion.

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